# Engineering Notes

*ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).*

## Method for Designing Approximate Optimal Controllers for Nonlinear Systems

James W. Austin*
*Rockwell International, Anaheim, Calif.*
and
Cornelius T. Leondes†
*University of California, Los Angeles, Calif.*

### Introduction

IN the application of optimal control theory to practical problems, one difficulty which arises is that the resulting control laws are difficult or expensive to mechanize. Hence, the desirability arises of developing simplified controllers which will, of course, be suboptimal. For example, approximations may have to be introduced into state estimation and error signal computation. In many cases, this has the effect of prespecifying the form of the controller while providing some adjustable parameters. Such approximations can result in significant degradation of the system performance, especially in stochastic problems. It is important, then, to find ways of minimizing the sensitivity to suboptimal controllers to, for example, unknown parameters in the system dynamics. One way to do this is to make use of free parameters which may exist in the controller. These parameters may be selected a priori (preflight) to reduce the sensitivity of the system to disturbances. In this Note, this problem is formulated as a problem in the calculus of variations. This results in an explicit minimization of a terminal error performance index for a nonlinear system, taking the form of a two-point boundary value problem. This provides a theoretical basis for the solution of this important practical problem, which has heretofore been treated on an ad hoc basis. Reference 1 gives an application of the method to computing target constants for an inertially guided ballistic missile or launch vehicle.

### Main Result

The method is a generalization of Sridhar's "specific optimal control problem"[2] to the stochastic case. "Specific" implies that the form of the control is prespecified, perhaps by hardware constraints, and that parameters therein are, in some sense, to be optimized. The development is limited to the case of unknown parameters in the system dynamics. In this Note, the Mayer form of performance index is assumed, although a more general form could easily be included.

The plant is assumed to be described by

$$\dot{x}=f(t,x,u,\eta) \qquad n\text{-dimensional} \tag{1}$$

Received June 30, 1978; revision received March 26, 1979.

Index categories: LV/M Guidance; Analytical and Numerical Methods.

*Member of the Technical Staff, System Analysis Unit.

†Professor of Engineering. Member AIAA.

where $\eta=(\eta_1,\ldots,\eta_q)^T$ is a vector of $q$ parameters which is unknown, but whose probability distribution function is assumed to be known. Assume that the form of the control law $u$ is prespecified, i.e.,

$$u(t)=\alpha(t,x,\xi)$$

is given, where

$$\xi=(\xi_1,\ldots,\xi_k)^T$$

is a constant vector to be optimized.

Then Eq. (1) may be written

$$\dot{x}=\phi(t,x,\xi,\eta)$$

Initial conditions: $x(t_0)=x_0$; terminal conditions: $\mathcal{E}x(T)=x_T$, where $\mathcal{E}$ denotes expected value and $x_T$ is the desired terminal condition.

$$\text{Performance index: } \min_{\xi} \underset{\eta}{\mathcal{E}}\, g[x(T)]$$

In words, it is desired to choose $\xi$ to minimize the expected value over $\eta$ of some given function of $x$ at the final time $T$. One method of solution is to augment the state vector $x$ with the vector $\xi$, i.e., let

$$z=(x,\xi)^T$$

$$\dot{\xi}=0 \quad \text{since } \xi \quad \text{is constant}$$

Then, the equivalent problem is

Plant: $\dot{z}=F(t,z,\eta)$

Performance index: $\min \mathcal{E}G[z(T)]$

Initial manifold: $Az(T_0)=x_0$; terminal manifold: $\mathcal{E}Az(T)=x_T$, since boundary conditions on $\xi$ are not specified.

The performance index will now be manipulated into a useful form. By definition,

$$\mathcal{E}_\eta G(z(T))=\int_{-\infty}^{\infty} G[z(T)]\,\mathrm{d}P(\eta) \tag{2}$$

where $P(\eta)$ is the probability distribution function (PDF) of $\eta$.

Assume that the performance index can be expressed as a quadratic criterion:

$$g(x(T))=(x(T)-\alpha)^T B(x(T)-\alpha)$$

or

$$G(z(T))=(Z(T)-\beta)^T C(z(T)-\beta)$$

$$=z(T)^T Cz(T)-\beta^T Cz(T)-z(T)^T C\beta+\beta^T C\beta \tag{3}$$

Substitute Eq. (3) into Eq. (2) and consider the term

$$\mathcal{E}z(T)^T C z(T)$$

$$=\mathcal{E}\Big(\int_{t_0}^T F(t,z,\eta)\,dt+z_0\Big)^T C\Big(\int_{t_0}^T F(t,z,\eta)\,dt+z_0\Big) \tag{4}$$

where $z_0=z(t_0)$.

Consider the term in Eq. (4):

$$\mathcal{E}\Big(\int_{t_0}^T F(t,z,\eta)\,dt\Big)^T C\Big(\int_{t_0}^T F(t,z,\eta)\,dt\Big) \tag{5}$$

Assume that the variations in $\eta$ are small enough that

$$F(t,z,\eta)=F(t,z,\eta_0)+\frac{\partial F}{\partial\eta}\Big|_{\eta_0}(\eta-\eta_0) \tag{6}$$

where $\eta_0=\mathcal{E}\eta$. Note that this does not imply linearization of the solution about a nominal trajectory; it will only be assumed in the evaluation of $\mathcal{E}G(z(T))$. Higher order terms can be included, if required, leading to higher moments in the result.

Then Eq. (5) becomes

$$\mathcal{E}\Big[\int_{t_0}^T\Big(F(t,z,\eta_0)+\frac{\partial F}{\partial\eta}\Delta\eta\Big)dt\Big]^T C\Big[\int_{t_0}^T\Big(F(t,z,\eta_0)+\frac{\partial F}{\partial\eta}\Delta\eta\Big)dt\Big]$$

$$=\mathcal{E}\Big(z_n(T)-z_0+\int_{t_0}^T\frac{\partial F}{\partial\eta}\Delta\eta\,dt\Big)^T C\Big(z_n(T)-z_0+\int_{t_0}^T\frac{\partial F}{\partial\eta}\Delta\eta\,dt\Big)$$

where $z_n(T)=z(T)$ when $\eta=\eta_0$;

$$=\mathcal{E}(z_n(T)-z_0)^T C(z_n(T)-z_0)$$

$$+\mathcal{E}\Big(\int_{t_0}^T\frac{\partial F}{\partial\eta}\Delta\eta\,dt\Big)^T C\Big(\int_{t_0}^T\frac{\partial F}{\partial\eta}\Delta\eta\,dt\Big)$$

since $\mathcal{E}\Delta\eta=0$;

$$=(z_n(T)-z_0)^T C(z_n(T)-z_0)$$

$$+\mathcal{E}\Big(\int_{t_0}^T\frac{\partial F}{\partial\eta}\Delta\eta\,dt\Big)^T C\Big(\int_{t_0}^T\frac{\partial F}{\partial\eta}\Delta\eta\,dt\Big) \tag{7}$$

since $z_n(T)=\mathcal{E}(z(T))$ is a consequence of assumption (6). Consider the term in Eq. (7):

$$\mathcal{E}\Big(\int_{t_0}^T\frac{\partial F}{\partial\eta}\Delta\eta\,dt\Big)^T C\Big(\int_{t_0}^T\frac{\partial F}{\partial\eta}\Delta\eta\,dt\Big)$$

$$=\mathrm{tr}\;C\mathcal{E}\Big(\int_{t_0}^T\frac{\partial F}{\partial\eta}\Delta\eta\,dt\Big)\Big(\int_{t_0}^T\frac{\partial F}{\partial\eta}\Delta\eta\,dt\Big)^T$$

by the identity $x^TAx=\mathrm{tr}\,A\,xx^T$;

$$=\mathrm{tr}\;C\mathcal{E}\int_{t_0}^T\int_{t_0}^T\frac{\partial F}{\partial\eta}\Delta\eta\Delta\eta^T\frac{\partial F}{\partial\eta}\,dt\,d\tau$$

$$=\mathrm{tr}\;C\int_{t_0}^T\int_{t_0}^T\frac{\partial F}{\partial\eta}N\frac{\partial F^T}{\partial\eta}\,dt\,d\tau$$

where $N=\mathrm{cov}\eta=\mathcal{E}(\eta-\eta_0)(\eta-\eta_0)^T$;

$$=\mathrm{tr}\;C\Big(\int_{t_0}^T\frac{\partial F}{\partial\eta}dt\Big)N\Big(\int_{t_0}^T\frac{\partial F}{\partial\eta}dt\Big)^T$$

$$=\mathrm{tr}\;C\Big(\frac{\partial}{\partial\eta}\int_{t_0}^T\dot{z}\,dt\Big)N\Big(\frac{\partial}{\partial\eta}\int_{t_0}^T\dot{z}\,dt\Big)^T$$

$$=\mathrm{tr}\;C\Big[\frac{\partial}{\partial\eta}(z(T)-z_0)\Big]N\Big[\frac{\partial}{\partial\eta}(z(T)-z_0)\Big]$$

$$=\mathrm{tr}\;C\frac{\partial z(T)}{\partial\eta}N\frac{\partial z(T)^T}{\partial\eta}$$

$=\psi(T)$ for notational convenience. Substituting back finally yields

$$\mathcal{E}G(z(T))=(z_n(T)-\beta)^T C(z_n(T)-\beta)$$
$$+2z_0^TCz_0+\mathrm{tr}\;C\frac{\partial z(T)}{\partial\eta}N\frac{\partial z(T)^T}{\partial\eta} \tag{8}$$

The first two terms in Eq. (8) are constant and need not be considered in the performance index. Hence, the equivalent problem is:

Plant: $\dot{z}=F(T,z,\eta_0)$ (9)

Performance index: $\min_{\xi}\;\mathrm{tr}\;C\dfrac{\partial z(T)}{\partial\eta}N\dfrac{\partial z(T)^T}{\partial\eta}$

Initial manifold: $Az(t_0)=x_0$

Terminal manifold: $Az(T)=x_T$ (10)

The solution of this Mayer problem will yield the optimal $\xi$. Physically, the sensitivity of the final state to the unknown parameters is minimized. Note that in the deterministic case, $N=0$ and the performance index $=0$ so the solution would not be unique.

Calculus of variations theory may now be applied by defining a vector of Lagrange multipliers $\lambda$ and a function $L$,

$$L=\lambda^T(\dot{z}-F(t,z,\eta_0))$$

Then the Euler-Lagrange necessary conditions for optimality,

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{z}}=\frac{\partial L}{\partial z}$$

lead to

$$\lambda^\circ=-\frac{\partial F^T}{\partial z}\lambda \tag{11}$$

The solution of this equation gives the minimizing function provided the minimum exists.[3]

Equations (9) and (11) are the system and adjoint equations, respectively. Equation (10) provides $2n$ boundary conditions on $x$. $2k$ boundary conditions on $\lambda$ are derived from the transversality condition, which for the Mayer problem is

$$\Big(\frac{\partial\psi}{\partial t}+L\Big)dt+\Big(\frac{\partial\psi}{\partial z}-\lambda\Big)^T dz=0$$

on the initial and terminal manifolds. Assume $T$ is fixed so $dt=0$. $dz$ is arbitrary as long as it stays on the manifold, i.e., $dx=0$, but $d\xi$ is arbitrary.

$$\therefore\lambda_i=\frac{\partial\psi}{\partial z_i}\Big|_{t=t_0,T}\qquad\text{for}\quad i=n+1,\dots,n+k$$

Thus, altogether there are $2(n+k)$ boundary conditions on $z$ and $\lambda$.

Equations (9) and (11) are solved preflight by a numerical technique such as the gradient method to obtain the optimal $\xi$ to be used in flight.

## References

[1]Austin, J.W., "Optimization and Estimation in Nonlinear Systems with Application to Inertial Guidance," Ph.D. in Engineering, University of California, Los Angeles, March 1978, pp. 1-50.

[2]Sridhar, R., unpublished lecture at California Institute of Technology, Pasadena, Calif., Feb. 1967.

[3]Tou, J.T., *Modern Control Theory,* McGraw-Hill, New York, 1964, p. 205.


# Stability Bounds for the Control of Large Space Structures


Suresh M. Joshi* and Nelson J. Groom†

*NASA Langley Research Center, Hampton, Va.*


## Introduction

STUDIES of future directions for the space program, such as Ref. 1, have identified a number of potentially important new space initiatives which will require large lightweight space structures with dimensions from one hundred to several thousands of meters. Control systems design for such structures is a challenging problem because the "plant" consists of infinite modes. Of these modes, only a finite number can be modeled and even fewer can be controlled due to practical limitations. Such tools as linear-quadratic Gaussian (LQG) theory[2] can be used for designing reduced-order controllers for such systems. Unfortunately, when going to reduced-order regulators and estimators, stability is no longer assured. The problem of control of large space structures (LSS) using the LQG approach then becomes one of developing new techniques for designing stable, robust, reduced-order regulators and estimators. Balas has discussed the stability problem of reduced-order regulators and estimators in terms of control and observation "spillover" in Ref. 3. The term "control spillover" was used to define that part of the feedback control which excites the uncontrolled (or residual) modes and "observation spillover" was used to define that part of the measurement contaminated by residual modes.

In this paper two different bounds are derived on spectral norms of control and observation spillover terms, any of which, if satisfied, assures asymptotic stability. The bounds are derived using Lyapunov methods. Numerical results are given for a long free-free beam.

## System Description and Definitions

The mathematical model of an LSS can be written as

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A_c & 0 \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x_c \\ x_r \end{bmatrix} + \begin{bmatrix} B_c \\ B_r \end{bmatrix} u + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{1}$$

where $x_c$ is the $n_c$-dimensional state vector for the controlled part of the system which consists of the rigid-body modes and some structural modes; $x_r$ is the $n_r$-dimensional state vector for the uncontrolled (or residual) modes; $A_c$, $A_r$, $B_c$, $B_r$ are appropriately dimensioned system and input matrices; $u$ is the $m$-dimensional input vector; and $v=(v_1^T, v_2^T)^T$ is the zero-mean Gaussian white noise input vector with covariance intensity matrix $V$. The $l$-dimensional observation vector is given by

$$y = C_c x_c + C_r x_r + w \tag{2}$$

where $C_c$ and $C_r$ are $l \times n_c$ and $l \times n_r$ matrices and $w$ is a zero-mean Gaussian white noise. Let the controller be given by

$$u = Gz \tag{3}$$

where $G$ is an $m \times p$ constant matrix, and $z$ is a $p$-dimensional compensator state vector

$$\dot{z} = A_{cz} z + B_z u + Ky \tag{4}$$

$A_{cz}$, $B_z$, and $K$ are appropriately dimensioned constant matrices. For example, an LQG controller designed for the $x_c$-system (truncating the residual system) has this structure. A regulator-observer system using pole placement would also have the same structure.

The resulting closed-loop equations, ignoring noise terms are

$$\begin{bmatrix} \dot{x}_c \\ \dot{z} \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A_c & B_c G & 0 \\ KC_c & A_{cz}+B_z G & 0 \\ 0 & 0 & A_r \end{bmatrix} \begin{bmatrix} x_c \\ z \\ x_r \end{bmatrix} + \begin{bmatrix} 0 \\ KC_r x_r \\ B_r G z \end{bmatrix} \tag{5}$$

This is compactly written as

$$\begin{bmatrix} \dot{x}_l \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A_l & 0 \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x_l \\ x_r \end{bmatrix} + \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix} \begin{bmatrix} x_l \\ x_r \end{bmatrix} \tag{6}$$

where

$$x_l = (x_c^T, z^T)^T \qquad A_l = \begin{bmatrix} A_c & B_c G \\ KC_c & A_{cz}+B_z G \end{bmatrix}$$

$$\alpha = \begin{bmatrix} 0 \\ KC_r \end{bmatrix} \qquad \beta = [0 \quad B_r G]$$

$$A = \begin{bmatrix} A_l & 0 \\ 0 & A_r \end{bmatrix} \quad \text{and } x = \begin{bmatrix} x_l \\ x_r \end{bmatrix}$$

Assume that $A_l$ and $A_r$ are strictly Hurwitz matrices. That is, the controlled part is stable, and the residual modes have some damping.

Let $(P,Q)$ be an ordered pair of positive definite symmetric $n \times n$ matrices $(n = n_c + n_r + p)$ satisfying

$$A^T P + PA = -Q \tag{7}$$

Let $(P_l, Q_l)$ and $(P_r, Q_r)$ be ordered pairs of $n_l \times n_l$ and $n_r \times n_r$ positive definite symmetric matrices $(n_l = n_c + p)$ satisfying

$$A_l^T P_l + P_l A_l = -Q_l \tag{8}$$

$$A_r^T P_r + P_r A_r = -Q_r \tag{9}$$

The spectral norm of a matrix $H$ is defined as

$$\|H\|_s = (\text{maximum eigenalue of } H^T H)^{1/2}$$

$\|\cdot\|$ denotes the Euclidian norm of a vector.


Received July 20, 1978; revision received Dec. 20, 1978. This paper is declared a work of the U.S. Government and therefore is in the public domain.

Index category: Spacecraft Dynamics and Control.


*Associate Professor (Research), Old Dominion University Research Foundation, Norfolk, Va.

†Aerospace Technologist.